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A recurrence method for a special class of continuous time linear programming problems

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Abstract This article studies a numerical solution method for a special class of continuous time linear programming problems denoted by (SP). We will present an efficient method for finding numerical solutions of (*SP*). The presented method is a discrete approximation algorithm, however, the main work of computing a numerical solution in our method is only to solve finite linear programming problems by using recurrence relations. By our constructive manner, we provide a computational procedure which would yield an error bound introduced by the numerical approximation. We also demonstrate that the searched approximate solutions weakly converge to an optimal solution. Some numerical examples are given to illustrate the provided procedure.

Keywords Continuous time linear programming problems · Infinite-dimensional linear programming problems

1 Introduction

Let $T > 0$ and $q \in \mathbb{N}$, and let $L_+^\infty[0, T]$ be the set of nonnegative real-valued, Lebesgue measurable, essentially bounded functions on the closed interval $[0, T]$. We consider an

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infinite-dimensional linear programming problem denoted by (*SP*) and defined as follows:

$$(SP): \text{maximize} \sum_{j=1}^{q} \int_0^T f_j(t)x_j(t)dt$$

$$\text{subject to} \sum_{j=1}^{q} \left[\beta_j x_j(t) - \int_0^t \gamma_j x_j(s)ds \right] \le g(t), \quad \forall\, t \in [0, T] \tag{1}$$

$$x_j(t) \in L_+^\infty[0, T], \quad \text{for } 1 \le j \le q,$$

where β_j and γ_j are given constants, $f_j(t) : [0, T] \to \mathbb{R}$ and $g(t) : [0, T] \to \mathbb{R}$ are given functions. $x_j(\cdot) : [0, T] \mapsto \mathbb{R}$ $(1 \le j \le q)$ is a decision variable. It is well known (refer to [3]) that the dual problem (DSP) of (SP) is defined as follows:

$$(DSP): \text{minimize} \int_0^T g(t)w(t)dt$$

$$\text{subject to } \beta_j\, w(t) - \int_t^T \gamma_j\, w(s)ds \ge f_j(t), \tag{2}$$

$$\forall\, 1 \le j \le q,\ t \in [0, T],$$

$$w(t) \in L_+^\infty[0, T],$$

where $w(\cdot) : [0, T] \mapsto \mathbb{R}$ is the decision variable.

(*SP*) is a special case of the so called continuous time linear programming problems (CLP), first introduced by Bellman [6] to model some production planning problems. The model of (CLP) has a wide range of applications (e.g., [6,8,21]), but is notoriously difficult to solve in general. In the literature, many researches have been proposed to consider (CLP). Studying the duality of (CLP), Grinold [12,13], Levinson [17] and Tyndall [22,23] have established strong duality theorems. Investigating a solution algorithm for (CLP), Anstreicher [5], Drews [9], Hartberger [14], Lehman [16], Perold [18] and Segers [20] have attempted to extend the simplex method to (CLP), however, the emerged theory is highly complex, and there remain substantial difficulties that make an implementation of the method unlikely to be successful. Studying a special case of (CLP), Anderson [1] introduced the separated continuous linear programs $(SCLP)$ to model job-shop scheduling problems. Since then many researches concerned with (CLP) have focused on $(SCLP)$ [2,4,10,19,24]. On the other hand, Buie and Abrham [7] proposed a discrete approximation method for finding numerical solutions of (CLP). As a solution technique, however, the provided method in [7] has some drawbacks. For instance, the searched numerical solutions may not be feasible; one cannot know how accurate the searched solution is; the termination criterion is not provided. Therefore, it would be useful to have a computational procedure which would yield bounds on the error introduced by the numerical approximation.

In this paper, we propose an efficient approximation method to approach the optimal value of (*SP*) by using recurrence relations. This method can be employed not only to easily solve (*SP*), but also to provide an error bound of the optimal value as well. Moreover, we also prove that our searched approximate solutions can converge weakly to an optimal solution of (*SP*).

For improving the readability, we define the notations $F(P)$ and $V(P)$ to be the feasible set and the optimal value of a linear programming problem (P), respectively. If $S_1, S_2 \subseteq \mathbb{R}$, then we denote by $C(S_1, S_2)$ the space of all continuous functions from S_1 to S_2. And the superscript " $\top$" denotes the transpose operation.

This paper is organized as follows. In Sect. 2, we develop a recurrence method for solving discretization problems (P_n) and (D_n) of (SP) and (DSP), respectively. In Sect. 3, we provide methods to construct approximate solutions for (SP) and (DSP). Moreover, we also establish an estimation for the error bounds of approximate values evaluated by the proposed method. In Sect. 4, we demonstrate that the searched approximate solutions weakly converge to an optimal solution. Finally, in Sect. 5, we provide some numerical examples to show the quality of the proposed error bound.

2 A recurrence method for solving discretization problems (P_n) and (D_n)

In the sequel of this paper we will make the following assumptions:

Assumption

(A1) $f_j(t) \in C([0,T], \mathbb{R})$ for all $1 \le j \le q$ and $g \in C([0,T], \mathbb{R}^+)$, where $\mathbb{R}^+$ is the set of all nonnegative real numbers.

(A2) $\beta_j > 0$ and $\gamma_j \ge 0$ for all $1 \le j \le q$.

To solve (SP) and (DSP), for each $n \in \mathbb{N}$, we let $P_{2^n} = \{0, \frac{1}{2^n}T, \frac{2}{2^n}T, \ldots, \frac{2^n-1}{2^n}T, T\}$ be a partition on $[0,T]$ into 2^n subintervals with equal length $\frac{T}{2^n}$. For $1 \le l \le 2^n$, let

$$b_l^{(n)} = min\left\{g(t) : t \in \left[\frac{l-1}{2^n}T, \frac{l}{2^n}T\right]\right\} \tag{3}$$

and

$$c_{jl}^{(n)} = min\left\{f_j(t) : t \in \left[\frac{l-1}{2^n}T, \frac{l}{2^n}T\right]\right\}, \tag{4}$$

for $1 \le j \le q$. We define step functions $g^{(n)}(t)$ and $f_j^{(n)}(t)$ as follows:

$$g^{(n)}(t) = \begin{cases} b_l^{(n)}, & \text{if } t \in \left[\frac{l-1}{2^n}T, \frac{l}{2^n}T\right), \\ b_{2^n}^{(n)}, & \text{if } t = T \end{cases} \tag{5}$$

and

$$f_j^{(n)}(t) = \begin{cases} c_{jl}^{(n)}, & \text{if } t \in \left[\frac{l-1}{2^n}T, \frac{l}{2^n}T\right), \\ c_{j2^n}^{(n)}, & \text{if } t = T, \end{cases} \tag{6}$$

where $1 \le l \le 2^n$. Consider the following programming problem:

$$(SP_n) : \text{maximize} \sum_{j=1}^{q} \int_0^T f_j^{(n)}(t)x_j(t)dt$$

$$\text{subject to} \sum_{j=1}^{q}\left[\beta_j x_j(t) - \int_0^t \gamma_j x_j(s)ds\right] \le g^{(n)}(t), \forall\, t \in [0,T],$$

$$x_j(t) \in L_+^{\infty}[0,T], \quad \text{for } 1 \le j \le q.$$

And its dual problem is defined as follows:

$$(DSP_n): \text{minimize} \int_0^T g^{(n)}(t)w(t)dt$$
$$\text{subject to } \beta_j\, w(t) - \int_t^T \gamma_j\, w(s)ds \geq f_j^{(n)}(t),$$
$$\forall\ 1 \leq j \leq q,\ \ t \in [0, T],\ \text{ and}$$
$$w(t) \in L_+^\infty[0, T].$$

Remark 1

(1a) Under assumption (A1), (*SP*) and (SP_n) are feasible for all $n \in \mathbb{N}$. Indeed, the zero vector functions is a common feasible solution of (*SP*) and (SP_n).

(1b) Under assumptions (A1) and (A2), (*DSP*) and (DSP_n) are feasible for all $n \in \mathbb{N}$. To see this, we choose $\alpha > 0$ such that $\alpha\beta_j \geq \gamma_j$ and $\alpha\beta_j \geq \max_{t\in[0,T]}\{f_j(t)\}$ for all $1 \leq j \leq q$. Define $\widetilde{w}(t) = \alpha e^{\alpha(T-t)}$, then $\widetilde{w}(t) \geq 0$. Besides, for $1 \leq j \leq q$ and $t \in [0, T]$, we have

$$\begin{aligned}
&\beta_j \widetilde{w}(t) - \int_t^T \gamma_j \widetilde{w}(s)ds \\
&= \beta_j \alpha e^{\alpha(T-t)} - \gamma_j \int_t^T \alpha e^{\alpha(T-s)}ds \\
&= \beta_j \alpha e^{\alpha(T-t)} + \gamma_j - \gamma_j e^{\alpha(T-t)} \\
&= \left[\alpha\beta_j - \gamma_j\right] e^{\alpha(T-t)} + \gamma_j \\
&\geq \alpha\beta_j - \gamma_j + \gamma_j \\
&= \alpha\beta_j \\
&\geq f_j(t) \\
&\geq f_j^{(n)}(t),
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence (*SP*) and (DSP_n) are feasible for all $n \in \mathbb{N}$.

(1c) It is well known (refer to [3]) that (*SP*) and (SP_n) have the weak duality property, that is, $V(SP) \leq V(DSP)$ and $V(SP_n) \leq V(DSP_n)$. Moreover, the strong duality for (*SP*) will be demonstrated by our constructive method, although the strong duality for general problem (CLP) has been established by Tyndall [22]. It is remarkable that the strong duality theorem has been extended to different versions by Grinold [12,13], Levinson [17] and Tyndall [23].

(1d) Because

$$g^{(1)}(t) \leq g^{(2)}(t) \leq \cdots \leq g^{(n)}(t) \leq \cdots \leq g(t),$$

and

$$f_j^{(1)}(t) \leq f_j^{(2)}(t) \leq \cdots \leq f_j^{(n)}(t) \leq \cdots \leq f_j(t),$$

for all $1 \le j \le q$ and $t \in [0, T]$, we have

$$F(SP_1) \subseteq F(SP_2) \subseteq \cdots \subseteq F(SP),$$

and

$$F(DSP_1) \supseteq F(DSP_2) \supseteq \cdots \supseteq F(DSP),$$

which implies

$$-\infty < V(SP_1) \le V(SP_2) \le \cdots \le V(SP) < \infty \tag{7}$$

and

$$-\infty < V(DSP_1) \le V(DSP_2) \le \cdots \le V(DSP) < \infty.$$

Hence

$$\lim_{n\to\infty} V(SP_n) \le V(SP)$$

and

$$\lim_{n\to\infty} (DSP_n) \le V(DSP). \tag{8}$$

Now we consider the finite dimensional linear programming problem which is due to (SP_n). For each $n \in \mathbb{N}$, let $b_l^{(n)}$ and $c_{jl}^{(n)}$ be defined as in (3) and (4). We define the following linear programming problem and use the convention that "empty sum," $\sum_1^0$ has the value zero.

$$\begin{aligned}(P_n):\ &\text{maximize} \sum_{j=1}^{q}\sum_{l=1}^{2^n} \frac{T}{2^n} c_{jl}^{(n)} x_{jl}\\ &\text{subject to} \sum_{j=1}^{q}\left[\beta_j x_{jl} - \frac{T}{2^n}\gamma_j \sum_{\alpha=1}^{l-1} x_{j\alpha}\right] \le b_l^{(n)},\ l = 1, 2, \ldots, 2^n,\\ &x_{jl} \ge 0,\ \ j = 1, 2, \ldots, q,\ \ l = 1, 2, \ldots, 2^n.\end{aligned}$$

The dual problem (D_n) of (P_n) is defined as follows:

$$\begin{aligned}(D_n):\ &\text{minimize} \sum_{l=1}^{2^n} \frac{T}{2^n} b_l^{(n)} w_l\\ &\text{subject to } \beta_j w_l - \frac{T}{2^n}\gamma_j \sum_{\alpha=l+1}^{2^n} w_\alpha \ge c_{jl}^{(n)},\ \ j = 1, 2, \ldots, q,\ \ l = 1, 2, \ldots, 2^n,\\ &w_l \ge 0,\ \ l = 1, 2, \ldots, 2^n,\end{aligned}$$

where "empty sum," $\sum_{2^n+1}^{2^n}$ has the value zero.

Remark 2

(2a) Under assumptions (A1) and (A2), the feasible set $F(P_n)$ is nonempty for all $n \in \mathbb{N}$, since the zero vector is a feasible solution of (P_n).

(2b) Under assumptions (A1) and (A2), for all $n \in \mathbb{N}$, the feasible set $F(D_n)$ is nonempty by the following Theorem 1.

(2c) By (2a),(2b) and the strong duality theorem of finite linear programming, under assumptions (A1) and (A2), both (P_n) and (D_n) have optimal solutions and $-\infty < V(P_n) = V(D_n) < \infty$.

The following results provide a recurrence method for solving (P_n) and (D_n). Let

$$L := \max_{1\leq j\leq q} \max_{0\leq t\leq T} \{f_j(t), 0\},$$
$$\sigma := \min_{1\leq j\leq q} \{\beta_j\}$$

and

$$\bar{\gamma} := \max_{1\leq j\leq q} \{\gamma_j\}.$$

Theorem 1 *Suppose that assumptions* (A1) *and* (A2) *hold. Let the vector* $\bar{w}^{(n)} = (\bar{w}_1^{(n)}, \ldots, \bar{w}_{2^n}^{(n)})^\top$ *be defined by*

$$\bar{w}_{2^n}^{(n)} := \max_{1\leq j\leq q} \left\{ \max\left\{ \frac{c_{j2^n}^{(n)}}{\beta_j}, 0 \right\} \right\},$$

and

$$\bar{w}_l^{(n)} := \max_{1\leq j\leq q} \left\{ \max\left\{ \frac{c_{jl}^{(n)} + \frac{T}{2^n}\gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)}}{\beta_j}, 0 \right\} \right\},$$

for $l = 2^n - 1, 2^n - 2, \ldots, 2, 1$. *Then*

(*i*) $\bar{w}^{(n)}$ *is an optimal solution of* (D_n).

(*ii*) *For* $1 \leq l \leq 2^n$

$$0 \leq \bar{w}_l^{(n)} \leq \frac{L}{\sigma}\left(1 + \frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n - l} \leq \frac{L}{\sigma}\left(1 + \frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n} \leq \frac{L}{\sigma} e^{\frac{T\bar{\gamma}}{\sigma}}. \tag{9}$$

Proof It is easy to check that $\bar{w}^{(n)}$ is feasible for (D_n). Now we claim that if $w^{(n)} = (w_1^{(n)}, \ldots, w_{2^n}^{(n)})^\top \in F(D_n)$ then $w_l^{(n)} \geq \bar{w}_l^{(n)}$ for all $1 \leq l \leq 2^n$. We prove it by induction. Obviously, $w_{2^n}^{(n)} \geq \bar{w}_{2^n}^{(n)}$ for all $w^{(n)} \in F(D_n)$. Suppose that $w_l^{(n)} \geq \bar{w}_l^{(n)}$ for all $l = k+1, k+2, \ldots, 2^n$. We will show that $w_k^{(n)} \geq \bar{w}_k^{(n)}$. Since $w^{(n)} \in F(D_n)$, $\beta_j w_k^{(n)} - \frac{T}{2^n}\gamma_j \sum_{\alpha=k+1}^{2^n} w_\alpha^{(n)} \geq c_{jk}^{(n)}$. This implies

$$w_k^{(n)} \geq \frac{c_{jk}^{(n)} + \frac{T}{2^n}\gamma_j \sum_{\alpha=k+1}^{2^n} w_\alpha^{(n)}}{\beta_j},$$

for all j. Hence

$$w_k^{(n)} \geq \max_{1\leq j\leq q} \left\{ \max\left\{ \frac{c_{jk}^{(n)} + \frac{T}{2^n}\gamma_j \sum_{\alpha=k+1}^{2^n} \bar{w}_\alpha^{(n)}}{\beta_j}, 0 \right\} \right\} = \bar{w}_k^{(n)}.$$

By induction on k, we show that our claim is valid. In view of (A1) and the fact that $b_l^{(n)} \geq 0$ for all l, we obtain that $\sum_{l=1}^{2^n} \frac{T}{2^n} b_l^{(n)} w_l^{(n)} \geq \sum_{l=1}^{2^n} \frac{T}{2^n} b_l^{(n)} \bar{w}_l^{(n)}$. Since $w^{(n)} \in F(D_n)$ is arbitrary, we see that $\bar{w}^{(n)}$ is an optimal solution of (D_n).

On the other hand, we assert that

$$\bar{w}_l^{(n)} \leq \frac{L}{\sigma}\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n-l},$$

for all $1 \leq l \leq 2^n$. It is obvious that

$$\bar{w}_{2^n}^{(n)} = \max_{1\leq j\leq q}\left\{\max\left\{\frac{c_{j2^n}^{(n)}}{\beta_j}, 0\right\}\right\} \leq \frac{L}{\sigma}$$

and

$$\bar{w}_{2^n-1}^{(n)} \leq \frac{L+\frac{T\bar{\gamma}L}{2^n\sigma}}{\sigma} = \frac{L}{\sigma}\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right).$$

Suppose that $\bar{w}_k^{(n)} \leq \frac{L}{\sigma}\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n-k}$ for all $k = l+1, \ldots, 2^n$. Then

$$\begin{aligned}
\bar{w}_l^{(n)} &= \max_{1\leq j\leq q}\left\{\max\left\{\frac{c_{jl}^{(n)} + \frac{T}{2^n}\gamma_j\sum_{\alpha=l+1}^{2^n}\bar{w}_\alpha^{(n)}}{\beta_j}, 0\right\}\right\} \\
&\leq \max_{1\leq j\leq q}\left\{\frac{c_{jl}^{(n)}}{\beta_j}, 0\right\} + \max_{1\leq j\leq q}\left\{\frac{\frac{T}{2^n}\gamma_j\sum_{\alpha=l+1}^{2^n}\bar{w}_\alpha^{(n)}}{\beta_j}\right\} \\
&\leq \frac{L}{\sigma} + \frac{T\bar{\gamma}}{2^n\sigma}\sum_{\alpha=l+1}^{2^n}\bar{w}_\alpha^{(n)} \\
&\leq \frac{L}{\sigma} + \frac{L}{\sigma}\left[\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n-l} - 1\right] \\
&= \frac{L}{\sigma}\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n-l}.
\end{aligned}$$

By induction the assertion is valid, and this implies that for all $1 \leq l \leq 2^n$

$$\bar{w}_l^{(n)} \leq \frac{L}{\sigma}\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n-l} \leq \frac{L}{\sigma}\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n} \leq \frac{L}{\sigma}e^{\frac{T\bar{\gamma}}{\sigma}},$$

since $\left(1+\frac{T\bar{\gamma}}{2^n\sigma}\right)^{2^n} \uparrow e^{\frac{T\bar{\gamma}}{\sigma}}$ as $n \to \infty$. We complete this proof. □

By the complementary slackness theorem, it is well known that $x^{(n)} \in F(P_n)$ and $w^{(n)} \in F(D_n)$ become an optimal solution pair if and only if $x^{(n)}$ and $w^{(n)}$ satisfy the following equations:

$$\left(b_l^{(n)} - \sum_{j=1}^{q}\left[\beta_j x_{jl}^{(n)} - \frac{T\gamma_j}{2^n}\sum_{\alpha=1}^{l-1}x_{j\alpha}^{(n)}\right]\right)w_l^{(n)} = 0, \quad \forall\ 1 \leq l \leq 2^n, \tag{10}$$

and

$$\left(\beta_j w_l^{(n)} - \frac{T\gamma_j}{2^n}\sum_{\alpha=l+1}^{2^n}w_\alpha^{(n)} - c_{jl}^{(n)}\right)x_{jl}^{(n)} = 0, \quad \forall\ 1 \leq j \leq q,\ 1 \leq l \leq 2^n. \tag{11}$$

Let us recall that an optimal solution $\bar{w}^{(n)}$ of (D_n) can be found easily by Theorem 1. Now we want to construct a feasible solution $\bar{x}^{(n)}$ of (P_n) which corresponds to the dual optimal solution $\bar{w}^{(n)}$ by the complementary slackness theorem.

Let

$$\Lambda := \{l : 1 \leq l \leq 2^n \text{ and } \bar{w}_l^{(n)} > 0\}, \tag{12}$$

and for $1 \leq l \leq 2^n$, $1 \leq j \leq q$

$$d_{jl} := \frac{c_{jl}^{(n)} + \frac{T}{2^n}\gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)}}{\beta_j}.$$

For $l \in \Lambda$, we let

$$\Lambda_{(l)} := \{\alpha : 1 \leq \alpha \leq l-1 \text{ and } \alpha \in \Lambda\}, \tag{13}$$

where $\Lambda_{(1)} = \emptyset$, and

$$j_{(l)} := \operatorname{argmax}_j\{d_{jl} : d_{jl} > 0\}. \tag{14}$$

(If more than one index j maximize d_{jl}, we let $j_{(l)}$ be the smallest such index.) Construct a $q \times 2^n$ matrix

$$\bar{x}^{(n)} = \left[\bar{x}_{jl}^{(n)}\right]_{q\times 2^n},$$

where

$$\bar{x}_{jl}^{(n)} := \begin{cases} 0 & \text{if } l \notin \Lambda, \\ 0 & \text{if } l \in \Lambda \text{ and } j \neq j_{(l)}, \\ \frac{b_l^{(n)} + \frac{T}{2^n}\sum_{\alpha\in\Lambda_{(l)}} \gamma_{j_{(\alpha)}} \bar{x}_{j_{(\alpha)}\alpha}^{(n)}}{\beta_{j_{(l)}}} & \text{if } l \in \Lambda \text{ and } j = j_{(l)}, \end{cases} \tag{15}$$

for $1 \leq j \leq q$ and $1 \leq l \leq 2^n$. We first show that $\bar{x}^{(n)} \in F(P_n)$. If $l \notin \Lambda$ then

$$b_l^{(n)} - \sum_{j=1}^{q}\left[\beta_j \bar{x}_{jl}^{(n)} - \frac{T}{2^n}\gamma_j \sum_{\alpha=1}^{l-1} \bar{x}_{j\alpha}^{(n)}\right] = b_l^{(n)} + \sum_{\alpha=1}^{l-1}\sum_{j=1}^{q} \frac{T}{2^n}\gamma_j \bar{x}_{j\alpha}^{(n)} \geq 0.$$

If $l \in \Lambda$ then

$$\begin{aligned} & b_l^{(n)} - \sum_{j=1}^{q}\left[\beta_j \bar{x}_{jl}^{(n)} - \frac{T}{2^n}\gamma_j \sum_{\alpha=1}^{l-1} \bar{x}_{j\alpha}^{(n)}\right] \\ &= b_l^{(n)} - \beta_{j_{(l)}} \bar{x}_{j_{(l)}l}^{(n)} + \sum_{\alpha=1}^{l-1}\sum_{j=1}^{q} \frac{T}{2^n}\gamma_j \bar{x}_{j\alpha}^{(n)} \\ &= b_l^{(n)} - \beta_{j_{(l)}} \bar{x}_{j_{(l)}l}^{(n)} + \sum_{\alpha\in\Lambda_{(l)}} \frac{T}{2^n}\gamma_{j_{(\alpha)}} \bar{x}_{j_{(\alpha)}\alpha}^{(n)} \\ &= 0. \end{aligned} \tag{16}$$

Hence $\bar{x}^{(n)} \in F(P_n)$. Now we assert that $\bar{x}^{(n)}$ and $\bar{w}^{(n)}$ satisfy equations (10) and (11). We verify this assertion by the following two cases.

Case 1. $\bar{w}_l^{(n)} = 0$. Then $\bar{x}_{jl}^{(n)} = 0$ for all $j = 1, 2, \ldots, q$. Hence

$$\left(b_l^{(n)} - \sum_{j=1}^{q} \left[\beta_j \bar{x}_{jl}^{(n)} - \frac{T}{2^n} \gamma_j \sum_{\alpha=1}^{l-1} \bar{x}_{j\alpha}^{(n)} \right] \right) \bar{w}_l^{(n)} = 0,$$

and

$$\left(\beta_j \bar{w}_l^{(n)} - \frac{T}{2^n} \gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)} - c_{jl}^{(n)} \right) \bar{x}_{jl}^{(n)} = 0, \text{ for all } j = 1, 2, \ldots, q.$$

Case 2. $\bar{w}_l^{(n)} > 0$. Then

$$\bar{w}_l^{(n)} = \frac{c_{j_{(l)}l}^{(n)} + \frac{T}{2^n} \gamma_{j_{(l)}} \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)}}{\beta_{j_{(l)}}},$$

and hence

$$\beta_{j_{(l)}} \bar{w}_l^{(n)} - \frac{T}{2^n} \gamma_{j_{(l)}} \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)} - c_{j_{(l)}l}^{(n)} = 0.$$

According to $\bar{x}_{jl}^{(n)} = 0$ for all $j \neq j_{(l)}$, we have

$$\left(\beta_j \bar{w}_l^{(n)} - \frac{T}{2^n} \gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)} - c_{jl}^{(n)} \right) \bar{x}_{jl}^{(n)} = 0, \ \forall\ 1 \leq j \leq q.$$

Note that $l \in \Lambda$, by (16), we have

$$b_l^{(n)} - \sum_{j=1}^{q} \left[\beta_j \bar{x}_{jl}^{(n)} - \frac{T}{2^n} \gamma_j \sum_{\alpha=1}^{l-1} \bar{x}_{j\alpha}^{(n)} \right] = 0.$$

Hence equations (10) and (11) hold. By Case 1 and 2, our assertion is valid, and hence $\bar{x}^{(n)}$ is an optimal solution of (P_n).

Based on the above discussion, we have an algorithm for solving (P_n) and (D_n).

Algorithm 1 Given $n, q \in \mathbb{N}$. Set $l = 2^n$ and $\Lambda = \emptyset$.

Step 1: Set $j = 1$.
Step 2: Compute $d_{jl} := \frac{c_{jl}^{(n)} + \frac{T}{2^n} \gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)}}{\beta_j}$ $\left(\sum_{\alpha=2^n+1}^{2^n} \bar{w}_\alpha^{(n)} := 0 \right)$.
Step 3: If $d_{jl} \leq 0$, set $\bar{x}_{jl}^{(n)} = 0$.
Step 4: If $j \neq q$, update $j \leftarrow j + 1$ and go to Step 2.
Step 5: If $d_{jl} \leq 0$ for all $j = 1, 2, \ldots, q$, set $\bar{w}_l^{(n)} = 0$.
Otherwise, update $\Lambda \leftarrow \Lambda \cup \{l\}$ and set $\bar{w}_l^{(n)} = \max\{d_{jl} : d_{jl} > 0\}$ and $\bar{x}_{jl}^{(n)} = 0$ for all $j \neq j_{(l)}$, where $j_{(l)}$ is defined in (14).
Step 6: If $l \neq 1$, update $l \leftarrow l - 1$ and go to Step 1.
Step 7: If $l \in \Lambda$, then find $\bar{x}_{j_{(l)}l}^{(n)}$ by solving the following recurrence relation:

$$\bar{x}_{j_{(l)}l}^{(n)} = \frac{b_l^{(n)} + \frac{T}{2^n} \sum_{\alpha \in \Lambda_{(l)}} \gamma_{j_{(\alpha)}} \bar{x}_{j_{(\alpha)}\alpha}^{(n)}}{\beta_{j_{(l)}}},$$

where $\Lambda_{(l)} = \{\alpha : 1 \le \alpha \le l-1 \text{ and } \alpha \in \Lambda\}$ and $\Lambda_{(1)} = \emptyset$.
Step 8: If $l = 2^n$, then STOP; otherwise update $l \leftarrow l+1$ and go to Step 7.

Moreover, the feasible set $F(P_n)$ is uniformly bounded. To see this, let

$$M := \max_{0\le t\le T} g(t).$$

Lemma 1 *Let* $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, \dots, x_{2^n}^{(n)}\right)^\top \in F(P_n)$, *where* $x_l^{(n)} = \left(x_{1l}^{(n)}, x_{2l}^{(n)}, \dots, x_{ql}^{(n)}\right)$ *and* $1 \le l \le 2^n$, *then*

$$0 \le x_{jl}^{(n)} \le \frac{M}{\sigma} e^{\frac{q\bar{\gamma}T}{\sigma}}, \tag{17}$$

for all $1 \le j \le q$ *and* $1 \le l \le 2^n$ *and* $n \in \mathbb{N}$.

Proof We claim that $x_{jl}^{(n)} \le \frac{M}{\sigma}\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{l-1}$ for all $1 \le j \le q$ and $1 \le l \le 2^n$. Since $\sum_{j=1}^q \beta_j x_{j1}^{(n)} \le b_1^{(n)}$ and $\beta_j > 0$, it follows that

$$x_{j1}^{(n)} \le \frac{b_1^{(n)}}{\beta_j} \le \frac{M}{\sigma}$$

for all j. Suppose that $x_{jl}^{(n)} \le \frac{M}{\sigma}\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{l-1}$ for $l = 1, 2, \dots, k-1$. Then

$$\sum_{l=1}^{k-1} x_{jl}^{(n)} \le \sum_{l=1}^{k-1} \frac{M}{\sigma}\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{l-1} = \frac{2^n M\left[\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{k-1} - 1\right]}{q\bar{\gamma}T} \tag{18}$$

for all j. Since $\sum_{j=1}^q \left[\beta_j x_{jk}^{(n)} - \frac{T\gamma_j}{2^n}\sum_{\alpha=1}^{k-1} x_{j\alpha}^{(n)}\right] \le b_k^{(n)}$ and by (18), we have that

$$\begin{aligned}\beta_j x_{jk}^{(n)} &\le b_k^{(n)} + \sum_{j=1}^q \frac{T\gamma_j}{2^n}\sum_{\alpha=1}^{k-1} x_{j\alpha}^{(n)} \\ &\le M + M\left[\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{k-1} - 1\right].\end{aligned}$$

This implies that $x_{jk}^{(n)} \le \frac{M}{\sigma}\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{k-1}$. Hence by induction we show that our claim is valid. Hence $x_{jl}^{(n)} \le \frac{M}{\sigma}\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{2^n}$ for all $1 \le j \le q$, $1 \le l \le 2^n$ and $n \in \mathbb{N}$. Since $\frac{M}{\sigma}\left(1 + \frac{q\bar{\gamma}T}{2^n\sigma}\right)^{2^n} \uparrow \frac{M}{\sigma}e^{\frac{q\bar{\gamma}T}{\sigma}}$ as $n \to \infty$, we get

$$0 \le x_{jl}^{(n)} \le \frac{M}{\sigma} e^{\frac{q\bar{\gamma}T}{\sigma}}.$$

The proof is complete. □

3 Approximate solutions of (*SP*) and (*DSP*)

In one of early papers, Tyndall [22] conjectured that the solutions to continuous time programming problems would be piecewise smooth functions under proper conditions. Jóhannesson and Hanson [15] confirmed this conjecture. However, it seems that no practical methods have been provided to find approximate solutions and values. In this section, we provide methods to construct approximate solutions for (*SP*) and (*DSP*) by virtue of the optimal solutions of (P_n) and (D_n).

Let $\bar{x}^{(n)}$ be defined in (15). Recall $\bar{x}^{(n)}$ is an optimal solution of (P_n). Define a step function $\hat{x}^{(n)}(\cdot) : [0, T] \mapsto \mathbb{R}^q$ as follows: $\hat{x}^{(n)}(t) = (\hat{x}_1^{(n)}(t), \hat{x}_2^{(n)}(t), \ldots, \hat{x}_q^{(n)}(t))^\top$, where for $1 \le j \le q$

$$\hat{x}_j^{(n)}(t) = \begin{cases} \bar{x}_{jl}^{(n)}, & \text{if } t \in [\frac{l-1}{2^n}T, \frac{l}{2^n}T) \text{ for some } 1 \le l \le 2^n \\ \bar{x}_{j2^n}^{(n)}, & \text{if } t = T. \end{cases} \tag{19}$$

Then we have the following result.

Lemma 2 *Let $\hat{x}^{(n)}(t)$ be defined as in (19). Then $\hat{x}^{(n)}(t) \in F(SP_n) \subseteq F(SP)$ for all $n \in \mathbb{N}$.*

Proof Since $\bar{x}^{(n)}$ is an optimal solution of (P_n),

$$\sum_{j=1}^{q}\left[\beta_j \bar{x}_{jl}^{(n)} - \frac{T\gamma_j}{2^n}\sum_{\alpha=1}^{l-1}\bar{x}_{j\alpha}^{(n)}\right] \le b_l^{(n)}, \tag{20}$$

for $1 \le l \le 2^n$. Consider the following two cases, we have that $\hat{x}^{(n)}(t) \in F(SP_n)$.

Case 1. $t \in \left[\frac{l-1}{2^n}T, \frac{l}{2^n}T\right)$, for some $1 \le l \le 2^n$. Then we have

$$\begin{aligned}
&\sum_{j=1}^{q}\left[\beta_j \hat{x}_j^{(n)}(t) - \int_0^t \gamma_j \hat{x}_j^{(n)}(s)ds\right] \\
&= \sum_{j=1}^{q}\left[\beta_j \hat{x}_j^{(n)}(t) - \sum_{\alpha=1}^{l-1}\int_{\frac{\alpha-1}{2^n}T}^{\frac{\alpha}{2^n}T} \gamma_j \hat{x}_j^{(n)}(s)ds - \int_{\frac{l-1}{2^n}T}^{t} \gamma_j \hat{x}_j^{(n)}(s)ds\right] \\
&\le \sum_{j=1}^{q}\left[\beta_j \bar{x}_{jl}^{(n)} - \frac{T\gamma_j}{2^n}\sum_{\alpha=1}^{l-1}\bar{x}_{j\alpha}^{(n)}\right]\left(\text{since } \int_{\frac{l-1}{2^n}T}^{t} \gamma_j \hat{x}_j^{(n)}(s)ds \ge 0\right) \\
&\le b_l^{(n)} \ (\text{by}(20)) \\
&= g^{(n)}(t).
\end{aligned}$$

Case 2. $t = T$. Then we have

$$\begin{aligned}
&\sum_{j=1}^{q}\left[\beta_j \hat{x}_j^{(n)}(T) - \int_0^T \gamma_j \hat{x}_j^{(n)}(s)ds\right] \\
&= \sum_{j=1}^{q}\left[\beta_j \bar{x}_{j2^n}^{(n)} - \sum_{\alpha=1}^{2^n}\frac{T\gamma_j}{2^n}\bar{x}_{j\alpha}^{(n)}\right] \\
&\le \sum_{j=1}^{q}\left[\beta_j \bar{x}_{j2^n}^{(n)} - \frac{T\gamma_j}{2^n}\sum_{\alpha=1}^{2^n-1}\bar{x}_{j\alpha}^{(n)}\right] \text{ (since } \gamma_j \ge 0) \\
&\le b_{2^n}^{(n)} \text{ (by(20))} \\
&= g^{(n)}(T).
\end{aligned}$$

We complete this proof. □

Moreover, it is obvious that

$$\sum_{j=1}^{q}\int_0^T f_j^{(n)}(t)\hat{x}_j^{(n)}(t)dt = \sum_{j=1}^{q}\sum_{l=1}^{2^n}\frac{T}{2^n}c_{jl}^{(n)}\bar{x}_{jl}^{(n)} = V(P_n). \tag{21}$$

Therefore, $V(SP_n) \ge V(P_n)$ and hence $V(DSP_n) \ge V(SP_n) \ge V(P_n) = V(D_n)$. Hence, by inequality (7), one can easily see that

$$V(DSP) \ge V(SP) \ge V(SP_n) \ge V(P_n) = V(D_n), \tag{22}$$

for all $n \in \mathbb{N}$.

Furthermore, we assert that $\lim_{n\to\infty} V(D_n) = V(DSP)$. To see this, we first need the following notations and lemma. Let

$$\epsilon_n := \max_{1\le j\le q}\sup_{t\in[0,T]}\{f_j(t) - f_j^{(n)}(t)\}, \tag{23}$$

$$\bar{\epsilon}_n := \sup_{t\in[0,T]}\{g(t) - g^{(n)}(t)\} \tag{24}$$

and

$$\rho := \max_{1\le j\le q}\left\{\frac{\gamma_j}{\beta_j}, \frac{1}{\beta_j}\right\}. \tag{25}$$

Let $\bar{w}^{(n)} = (\bar{w}_1^{(n)}, \bar{w}_2^{(n)}, \ldots, \bar{w}_{2^n}^{(n)})^\top$ be the optimal solution of (D_n) defined as in Theorem 1. Define a function $\hat{w}^{(n)}(t) : [0, T] \mapsto \mathbb{R}$ as follows:

$$\hat{w}^{(n)}(t) = \begin{cases} \bar{w}_l^{(n)} + \delta_{2^n}\rho e^{\rho(T-t)}, & \text{if } \frac{l-1}{2^n}T \le t < \frac{l}{2^n}T \text{ for some } 1 \le l \le 2^n, \\ \bar{w}_{2^n}^{(n)} + \delta_{2^n}\rho, & \text{if } t = T, \end{cases} \tag{26}$$

where

$$\delta_{2^n} := \max_{1\le l\le 2^n}\left\{\frac{T}{2^n}\bar{w}_l^{(n)}\right\}. \tag{27}$$

Moreover, define

$$\widetilde{w}^{(n)}(t) = \hat{w}^{(n)}(t) + \epsilon_n\rho e^{\rho(T-t)} \tag{28}$$

for all $t \in [0, T]$, where ϵ_n is defined as in (23). It can be shown by the following lemma that $\hat{w}^{(n)}(t)$ and $\widetilde{w}^{(n)}(t)$ are feasible solutions of (DSP_n) and (DSP), respectively.

Lemma 3 *Suppose that assumptions* (A1) *and* (A2) *hold. Let* $\hat{w}^{(n)}(t)$ *and* $\widetilde{w}^{(n)}(t)$ *be defined as above. Then*

(i) $\hat{w}^{(n)}(t) \in F(DSP_n)$ *and*

$$0 \leq \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt - V(D_n) \leq \delta_{2^n} \int_0^T \rho e^{\rho(T-t)} g(t)dt. \tag{29}$$

(ii) $\widetilde{w}^{(n)}(t) \in F(DSP)$ *and*

$$0 \leq \int_0^T g(t)\widetilde{w}^{(n)}(t)dt - \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt \leq \bar{\epsilon}_n \int_0^T \hat{w}^{(n)}(t)dt + \epsilon_n \int_0^T \rho e^{\rho(T-t)} g(t)dt, \tag{30}$$

where ϵ_n *and* $\bar{\epsilon}_n$ *are defined as in* (23) *and* (24)*, respectively.*

Proof

(i). We verify that $\hat{w}^{(n)}(t) \in F(DSP_n)$ by the following two cases.

Case 1. $t \in \left[\frac{l-1}{2^n}T, \frac{l}{2^n}T\right)$ for some $1 \leq l \leq 2^n$. Then, for $1 \leq j \leq q$,

$$\begin{aligned}
&\beta_j \hat{w}^{(n)}(t) - \int_t^T \gamma_j \hat{w}^{(n)}(s)ds \\
&= \beta_j \hat{w}^{(n)}(t) - \left[\int_t^{\frac{l}{2^n}T} \gamma_j \hat{w}^{(n)}(s)ds + \sum_{\alpha=l+1}^{2^n} \int_{\frac{\alpha-1}{2^n}T}^{\frac{\alpha}{2^n}T} \gamma_j \hat{w}^{(n)}(s)ds \right] \\
&= \beta_j \left(\bar{w}_l^{(n)} + \delta_{2^n} \rho e^{\rho(T-t)} \right) - \left[\left(\frac{l}{2^n}T - t \right) \bar{w}_l^{(n)} \gamma_j + \gamma_j \delta_{2^n} \int_t^{\frac{l}{2^n}T} \rho e^{\rho(T-s)} ds \right. \\
&\quad \left. + \frac{T}{2^n} \gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)} + \gamma_j \delta_{2^n} \sum_{\alpha=l+1}^{2^n} \int_{\frac{\alpha-1}{2^n}T}^{\frac{\alpha}{2^n}T} \rho e^{\rho(T-s)} ds \right] \\
&= \beta_j \bar{w}_l^{(n)} + \beta_j \delta_{2^n} \rho e^{\rho(T-t)} - \left(\frac{l}{2^n}T - t \right) \gamma_j \bar{w}_l^{(n)} - \frac{T}{2^n} \gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)} \\
&\quad - \gamma_j \delta_{2^n} \int_t^T \rho e^{\rho(T-s)} ds
\end{aligned}$$

$$= \beta_j \bar{w}_l^{(n)} - \frac{T}{2^n}\gamma_j \sum_{\alpha=l+1}^{2^n} \bar{w}_\alpha^{(n)} + \beta_j \delta_{2^n} \rho e^{\rho(T-t)} - \left(\frac{l}{2^n}T - t\right)\gamma_j \bar{w}_l^{(n)}$$
$$- \gamma_j \delta_{2^n}(e^{\rho(T-t)} - 1)$$
$$\geq c_{jl}^{(n)} + \left(\rho\beta_j - \gamma_j\right)\delta_{2^n} e^{\rho(T-t)} - \left(\frac{l}{2^n}T - t\right)\gamma_j \bar{w}_l^{(n)} + \gamma_j \delta_{2^n}$$
$$\geq c_{jl}^{(n)} + \left(\rho\beta_j - \gamma_j\right)\delta_{2^n} - \frac{T}{2^n}\gamma_j \bar{w}_l^{(n)} + \gamma_j \delta_{2^n}$$
$$\geq c_{jl}^{(n)} + \left(\rho\beta_j - \gamma_j\right)\delta_{2^n} \quad \left(\text{since } \delta_{2^n} \geq \frac{T}{2^n}\bar{w}_l^{(n)}\right)$$
$$\geq c_{jl}^{(n)} = f_j^{(n)}(t).$$

Case 2. $t = T$. Then for $1 \leq j \leq q$,

$$\beta_j \hat{w}^{(n)}(T) - \int_T^T \gamma_j \hat{w}^{(n)}(s)ds$$
$$= \beta_j \hat{w}^{(n)}(T) = \beta_j(\bar{w}_{2^n}^{(n)} + \delta_{2^n}\rho)$$
$$\geq \beta_j \bar{w}_{2^n}^{(n)} \geq c_{j2^n}^{(n)} = f_j^{(n)}(T).$$

Hence $\hat{w}^{(n)}(t) \in F(DSP_n)$.

Moreover, we observe that

$$\int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt$$
$$= \sum_{l=1}^{2^n} b_l^{(n)} \int_{\frac{l-1}{2^n}T}^{\frac{l}{2^n}T} \bar{w}_l^{(n)} dt + \delta_{2^n} \int_0^T \rho e^{\rho(T-t)} g^{(n)}(t)dt$$
$$= \sum_{l=1}^{2^n} b_l^{(n)} \frac{T}{2^n} \bar{w}_l^{(n)} + \delta_{2^n} \int_0^T \rho e^{\rho(T-t)} g^{(n)}(t)dt$$
$$= V(D_n) + \delta_{2^n} \int_0^T \rho e^{\rho(T-t)} g^{(n)}(t)dt, \tag{31}$$

which implies $0 \leq \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt - V(D_n) \leq \delta_{2^n} \int_0^T \rho e^{\rho(T-t)} g(t)dt$, since $g^{(n)}(t) \leq g(t)$.

(ii). Observe that $\widetilde{w}^{(n)}(t) \geq \hat{w}^{(n)}(t) \geq 0$ for all $t \in [0, T]$. For $1 \leq j \leq q$ we have

$$\beta_j \widetilde{w}^{(n)}(t) - \int_t^T \gamma_j \widetilde{w}^{(n)}(s)ds$$

$$
\begin{aligned}
&= \beta_j \hat{w}^{(n)}(t) - \int_t^T \gamma_j \hat{w}^{(n)}(s)ds + \epsilon_n \left[\beta_j \rho e^{\rho(T-t)} - \int_t^T \gamma_j \rho e^{\rho(T-s)} ds \right] \\
&\geq f_j^{(n)}(t) + \epsilon_n \left[\beta_j \rho e^{\rho(T-t)} + \gamma_j - \gamma_j e^{\rho(T-t)} \right] \\
&= f_j^{(n)}(t) + \epsilon_n \left[(\rho\beta_j - \gamma_j) e^{\rho(T-t)} + \gamma_j \right] \\
&\geq f_j^{(n)}(t) + \epsilon_n \left[\rho\beta_j - \gamma_j + \gamma_j \right] \ (\text{since } \rho\beta_j \geq \gamma_j) \\
&= f_j^{(n)}(t) + \epsilon_n \rho\beta_j \\
&\geq f_j^{(n)}(t) + \epsilon_n \ (\text{since } \rho\beta_j \geq 1) \\
&\geq f_j(t) \text{ for all } t \in [0, T].
\end{aligned}
$$

Hence $\widetilde{w}^{(n)}(t) \in F(DSP)$.
Since $\widetilde{w}^{(n)}(t) \geq \hat{w}^{(n)}(t) \geq 0$ and $g(t) \geq g^{(n)}(t) \geq 0$, we have

$$
\begin{aligned}
0 &\leq \int_0^T g(t)\widetilde{w}^{(n)}(t)dt - \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt \\
&= \int_0^T [g(t) - g^{(n)}(t)]\hat{w}^{(n)}(t)dt + \epsilon_n \int_0^T g(t)\rho e^{\rho(T-t)}dt \\
&\leq \bar{\epsilon}_n \int_0^T \hat{w}^{(n)}(t)dt + \epsilon_n \int_0^T \rho e^{\rho(T-t)} g(t)dt.
\end{aligned}
$$

We complete this proof. □

Note that, by (30), we have

$$
\begin{aligned}
V(DSP) &- \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt \\
&\leq \int_0^T g(t)\widetilde{w}^{(n)}(t)dt - \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt \\
&\leq \bar{\epsilon}_n \int_0^T \hat{w}^{(n)}(t)dt + \epsilon_n \int_0^T g(t)\rho e^{\rho(T-t)}dt. \qquad (32)
\end{aligned}
$$

Hence, by (22), (29) and (32), we have

$$
\begin{aligned}
0 &\leq V(DSP) - V(D_n) \\
&= V(DSP) - \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt + \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt - V(D_n) \\
&\leq \bar{\epsilon}_n \int_0^T \hat{w}^{(n)}(t)dt + \epsilon_n \int_0^T \rho e^{\rho(T-t)} g(t)dt + \delta_{2^n} \int_0^T \rho e^{\rho(T-t)} g(t)dt
\end{aligned}
$$

$$= \bar{\epsilon}_n \int_0^T \hat{w}^{(n)}(t)dt + (\epsilon_n + \delta_{2^n}) \int_0^T \rho e^{\rho(T-t)} g(t)dt$$

$$= \bar{\epsilon}_n \left[\sum_{l=1}^{2^n} \frac{T}{2^n} \bar{w}_l^{(n)} + \delta_{2^n}(e^{\rho T} - 1) \right] + (\epsilon_n + \delta_{2^n}) \int_0^T \rho e^{\rho(T-t)} g(t)dt$$

$$\leq \bar{\epsilon}_n \left[2^n \delta_{2^n} + \delta_{2^n}(e^{\rho T} - 1) \right] + (\epsilon_n + \delta_{2^n}) \int_0^T \rho e^{\rho(T-t)} g(t)dt \quad \text{(by(27))}$$

$$= \bar{\epsilon}_n \delta_{2^n}(2^n + e^{\rho T} - 1) + (\epsilon_n + \delta_{2^n}) \int_0^T \rho e^{\rho(T-t)} g(t)dt. \tag{33}$$

Note that $\epsilon_n \to 0$ and $\bar{\epsilon}_n \to 0$ as $n \to \infty$, since $f_j(t)$ and $g(t)$ are uniformly continuous on $[0, T]$. Accordingly, by (27) and Theorem 1-(ii), $\delta_{2^n} \to 0$ and

$$\bar{\epsilon}_n \delta_{2^n} 2^n \leq \bar{\epsilon}_n T \frac{L}{\sigma} e^{\frac{T\bar{\gamma}}{\sigma}} \to 0 \text{ } as \text{ } n \to \infty.$$

Thus we have $\lim_{n\to\infty} V(D_n) = V(DSP)$.

Based on the above discussion, we have the following result which provides the estimation for the error between $V(DSP)$ and $V(D_n)$ and the error between $V(SP)$ and $V(P_n)$.

Theorem 2 *Suppose that assumptions* (A1) *and* (A2) *hold. Then the sequence* $\{V(D_n)\}$ *is convergent to* $V(DSP)$. *Moreover, we have*

$$0 \leq V(DSP) - V(D_n) \leq \varepsilon_n,$$

where

$$\varepsilon_n := \bar{\epsilon}_n \delta_{2^n}(2^n + e^{\rho T} - 1) + (\epsilon_n + \delta_{2^n}) \int_0^T \rho e^{\rho(T-t)} g(t)dt, \tag{34}$$

ϵ_n, $\bar{\epsilon}_n$ *and* δ_{2^n} *are defined as in* (23), (24) *and* (27), *respectively.*

Note that, by inequality (22) and Theorem 2, we have

$$V(DSP) \geq V(SP) \geq \lim_{n\to\infty} V(D_n) = V(DSP).$$

Therefore, $V(DSP) = V(SP) = \lim_{n\to\infty} V(D_n) = \lim_{n\to\infty} V(P_n)$, and

$$0 \leq V(SP) - V(P_n) \leq \varepsilon_n,$$

where ε_n is defined by (34). Moreover, we can establish the estimation for the error bound of objective values of approximate solutions $\hat{x}^{(n)}(t)$ and $\widetilde{w}^{(n)}(t)$ to (*SP*) and (*DSP*), respectively.

Theorem 3 *Suppose that assumptions* (A1) *and* (A2) *hold. Let* $\hat{x}^{(n)}(t)$ *and* $\widetilde{w}^{(n)}(t)$ *be defined as in* (19) *and* (28), *respectively. Then the error between the optimal value of* (*SP*) *and the objective value of* $\hat{x}^{(n)}(t)$ *and the error between the optimal value of* (*DSP*) *and the objective value of* $\widetilde{w}^{(n)}(t)$ *are both less than or equal to* ε_n.

Proof By Lemma 2, $\hat{x}^{(n)}(t) \in F(SP)$. Since $f_j^{(n)}(t) \le f_j(t)$ for every j and

$$\sum_{j=1}^{q} \int_0^T f_j^{(n)}(t)\hat{x}_j^{(n)}(t)dt = \sum_{j=1}^{q} \sum_{l=1}^{2^n} \frac{T}{2^n} c_{jl}^{(n)} \bar{x}_{jl}^{(n)} = V(P_n) = V(D_n),$$

we have

$$\begin{aligned} 0 &\le V(SP) - \sum_{j=1}^{q} \int_0^T f_j(t)\hat{x}_j^{(n)}(t)dt \\ &\le V(SP) - \sum_{j=1}^{q} \int_0^T f_j^{(n)}(t)\hat{x}_j^{(n)}(t)dt \\ &= V(DSP) - V(D_n) \\ &\le \varepsilon_n, \end{aligned}$$

by Theorem 2.

On the other hand, since $V(D_n) \le V(DSP)$, we have

$$\begin{aligned} 0 &\le \int_0^T g(t)\widetilde{w}^{(n)}(t)dt - V(DSP) \\ &\le \int_0^T g(t)\widetilde{w}^{(n)}(t)dt - V(D_n) \\ &= \int_0^T g(t)\widetilde{w}^{(n)}(t)dt - \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt + \int_0^T g^{(n)}(t)\hat{w}^{(n)}(t)dt - V(D_n) \\ &\le \bar{\epsilon}_n \int_0^T \hat{w}^{(n)}(t)dt + (\epsilon_n + \delta_{2^n}) \int_0^T \rho e^{\rho(T-t)} g(t)dt \quad \text{(by (29) and (30))} \\ &\le \varepsilon_n \quad \text{(by (33))}. \end{aligned}$$

We complete this proof. □

4 Algorithm and convergence of approximate solutions

We summarize the preceding discussions to form the following solution procedure for finding the approximate solutions of (*SP*) and (*DSP*).

Algorithm 2 Let δ be the accuracy of tolerance and an initial number $n_0 \in \mathbb{N}$ be given.

Step 1: Set $n \leftarrow n_0$.
Step 2: Calculate $\bar{x}^{(n)}$ and $\bar{w}^{(n)}$ by Algorithm 1. Compute the error bound ε_n defined as in (34).
Step 3: If $\varepsilon_n \le \delta$, then STOP. By (19) and (28) , construct $\hat{x}^{(n)}(t)$ and $\widetilde{w}^{(n)}(t)$ as the approximate solutions of (*SP*) and (*DSP*), respectively.
Otherwise, update $n \leftarrow n+1$, and go to Step 2.

In what follows, we will demonstrate the convergent properties of the sequences $\{\hat{x}^{(n)}(t)\}$ and $\{\widetilde{w}^{(n)}(t)\}$ derived by Algorithm 2.

Let $L^1[0, T]$ be the family of equivalence classes of real-valued Lebesgue measurable functions on $[0, T]$ with finite L^1 norm. The dual space of the separable Banach space $L^1[0, T]$ can be identified with $L^\infty[0, T]$. An important property enjoyed by the dual of a separable Banach space is weak-star sequential compactness for sets bounded in the strong topology.

By [11, Theorem 4.12.3] and [17, Lemma 2.1], we have the following useful lemma.

Lemma 4 *Let $\lambda_n \in L^\infty[0, T]$. If there exists a constant $\kappa > 0$ such that $\|\lambda_n\|_\infty \leq \kappa$ for $n = 1, 2, \ldots$. Then*

(i) *there exist $\lambda \in L^\infty[0, T]$ and a subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \to \lambda$ (weak*), that is,*

$$\int_0^T \lambda_{n_k}(t)h(t)dt \to \int_0^T \lambda(t)h(t)dt \quad \text{for all } h(t) \in L^1[0, T];$$

(ii) *we have*

$$\lambda(t) \leq \limsup_{n_k \to \infty} \lambda_{n_k}(t) \quad \text{for almost all } t \in [0, T]$$

and

$$\lambda(t) \geq \liminf_{n_k \to \infty} \lambda_{n_k}(t) \quad \text{for almost all } t \in [0, T].$$

Remark 3 Note that if $\lambda_n(t) \geq 0$ for all $t \in [0, T]$ and $\lambda_n \to \lambda$ (weak*), then $\lambda(t) \geq 0$ for almost all $t \in [0, T]$ by Lemma 4 (ii).

We also note that both the feasible domains of (*SP*) and (*DSP*) are in $L^\infty[0, T]$, which is normally regarded as a family of equivalence classes, however, the original formulations require that the feasible solutions must satisfy the constraints for all t not only for almost everywhere. In this section, based on our constructed method for approximate solutions of (*SP*) and (*DSP*), we can show that there exist two functions $\hat{x}^\star(t)$ and $\widetilde{w}^\star(t)$, as shown in Theorem 4, such that they have the same objective value and satisfy the constraints of (*SP*) and (*DSP*), respectively, for all t not only for almost everywhere. Hence, by the weak duality property shown in Remark (1c), they are optimal solutions of (*SP*) and (*DSP*), respectively.

To see this, the following Lemma 5 developed by Tyndall [22, Lemma 5] is needed. Let

$$F_L(SP) := \{x(t) = (x_1(t), \ldots, x_q(t))^\top \mid x_j(t) \in L_+^\infty[0, T] \ (1 \leq j \leq q) \text{ and}$$
$$\sum_{j=1}^q \left(\beta_j x_j(t) - \int_0^t \gamma_j x_j(s)ds \right) \leq g(t) \text{ for almost all } t \in [0, T]\}$$

and

$$F_L(DSP) := \{w(t) \mid w(t) \in L_+^\infty[0, T] \text{ and } \beta_j\, w(t) - \int_t^T \gamma_j\, w(s)ds \geq f_j(t) \ (1 \leq j \leq q)$$
$$\text{for almost all } t \in [0, T]\}.$$

Lemma 5 *Given $x(t) = (x_1(t), \ldots, x_q(t))^\top \in F_L(SP)$ and $w(t) \in F_L(DSP)$. Then there exist $\hat{x}(t) = (\hat{x}_1(t), \ldots, \hat{x}_q(t))^\top \in F(SP)$ and $\tilde{w}(t) \in F(DSP)$ such that $x_j(t) = \hat{x}_j(t)$ $(1 \leq j \leq q)$ and $w(t) = \tilde{w}(t)$ for almost all $t \in [0, T]$.*

Let $\hat{x}^{(n)}(t)$ and $\widetilde{w}^{(n)}(t)$ be defined as in (19) and (28), respectively. Then we have the following result.

Theorem 4 *Suppose that assumptions* (A1) *and* (A2) *hold. Then there exist subsequences* $\{\hat{x}^{(n_k)}(t)\}$ *and* $\{\widetilde{w}^{(n_k)}(t)\}$ *such that* $\hat{x}^{(n_k)} \to \hat{x}^\star$ $(weak^*)$ *and* $\widetilde{w}^{(n_k)} \to \widetilde{w}^\star$ $(weak^*)$. *Moreover,* $\hat{x}^\star(t)$ *and* $\widetilde{w}^\star(t)$ *are the optimal solutions of* (*SP*) *and* (*DSP*)*, respectively.*

Proof Note that, by Theorem 1 and Lemma 1, there exists a constant $\kappa > 0$ such that $\|\hat{x}_j^{(n)}\|_\infty < \kappa$ and $\|\widetilde{w}^{(n)}\|_\infty < \kappa$ for all n and $1 \le j \le q$. By Lemma 4, there exist $\hat{x}_j(t) \in L^\infty[0,T]$, $\widetilde{w}(t) \in L^\infty[0,T]$ and subsequences $\{\hat{x}_j^{(n_k)}(t)\}$ and $\{\widetilde{w}^{(n_k)}(t)\}$ such that

$$\hat{x}_j^{(n_k)} \to \hat{x}_j \ (weak^*) \quad \text{for all } 1 \le j \le q \tag{35}$$

and

$$\widetilde{w}^{(n_k)} \to \widetilde{w} \ (weak^*). \tag{36}$$

Besides, we have

$$\hat{x}_j(t) \le \limsup_{n_k \to \infty} \hat{x}_j^{(n_k)}(t) \quad \text{for almost all } t \in [0,T], \tag{37}$$

and

$$\widetilde{w}(t) \ge \liminf_{n_k \to \infty} \widetilde{w}^{(n_k)}(t) \quad \text{for almost all } t \in [0,T]. \tag{38}$$

Since $\hat{x}^{(n_k)}(t) \in F(SP)$ and $\widetilde{w}^{(n_k)}(t) \in F(DSP)$, we have for all $t \in [0,T]$

$$\sum_{j=1}^{q} \left[\beta_j \hat{x}_j^{(n_k)}(t) - \int_0^t \gamma_j \hat{x}_j^{(n_k)}(s)ds \right] \le g(t), \tag{39}$$

$$\hat{x}_j^{(n_k)}(t) \ge 0$$

and for all $t \in [0,T]$

$$\beta_j \widetilde{w}^{(n_k)}(t) - \int_t^T \gamma_j \widetilde{w}^{(n_k)}(s)ds \ge f_j(t) \ (1 \le j \le q), \tag{40}$$

$$\widetilde{w}^{(n_k)}(t) \ge 0.$$

Since $\hat{x}_j^{(n_k)}(t) \ge 0$ and $\widetilde{w}^{(n_k)}(t) \ge 0$, it follows, by (35), (36) and Remark 3, that $\hat{x}_j(t) \ge 0$ and $\widetilde{w}(t) \ge 0$ for almost all $t \in [0,T]$. From (39) and (40), by taking the limit superior and inferior, we obtain, for almost all $t \in [0,T]$,

$$\begin{aligned}\sum_{j=1}^{q}\beta_j\hat{x}_j(t) &\le \limsup_{n_k\to\infty}\sum_{j=1}^{q}\beta_j\hat{x}_j^{(n_k)}(t)\ \text{(by (37))}\\ &\le \limsup_{n_k\to\infty}\sum_{j=1}^{q}\int_0^t\gamma_j\hat{x}_j^{(n_k)}(s)ds+g(t)\ \text{(by (39))}\\ &=\sum_{j=1}^{q}\int_0^t\gamma_j\hat{x}_j(s)ds+g(t)\ \text{(by (35))}\end{aligned}$$

and

$$\begin{aligned}\beta_j\widetilde{w}(t) &\ge \liminf_{n_k\to\infty}\beta_j\widetilde{w}^{(n_k)}(t)\ \text{(by (38))}\\ &\ge \liminf_{n_k\to\infty}\int_t^T\gamma_j\widetilde{w}^{(n_k)}(s)ds+f_j(t)\ \text{(by (40))}\\ &=\int_t^T\gamma_j\widetilde{w}(s)ds+f_j(t)\quad\text{for all } j\ \text{(by (36))}.\end{aligned}$$

Hence $\hat{x}(t)=(\hat{x}_1(t),\ldots,\hat{x}_q(t))^\top\in F_L(SP)$ and $\widetilde{w}(t)\in F_L(DSP)$. By Lemma 5, there exist $\hat{x}^\star(t)=(\hat{x}_1^\star(t),\ldots,\hat{x}_q^\star(t))^\top\in F(SP)$ and $\widetilde{w}^\star(t)\in F(DSP)$ such that $\hat{x}_j^\star(t)=\hat{x}_j(t)$ $(1\le j\le q)$ and $\widetilde{w}^\star(t)=\widetilde{w}(t)$ for almost all $t\in[0,T]$. Thus, by (35) and (36), we have

$$\hat{x}^{(n_k)}\to\hat{x}^\star\ (weak^*)\ \text{and}\ \widetilde{w}^{(n_k)}\to\widetilde{w}^\star\ (weak^*).$$

Next, we will prove that $\hat{x}^\star(t)$ and $\widetilde{w}^\star(t)$ are the optimal solutions of (*SP*) and (*DSP*), respectively. We observe that

$$\begin{aligned}&\int_0^T\sum_{j=1}^{q}f_j(t)\hat{x}_j^{(n_k)}(t)dt\\ &=\sum_{j=1}^{q}\int_0^T\left[f_j(t)-f_j^{(n_k)}(t)\right]\hat{x}_j^{(n_k)}(t)dt+\sum_{j=1}^{q}\int_0^T f_j{}^{(n_k)}(t)\hat{x}_j^{(n_k)}(t)dt\\ &=\sum_{j=1}^{q}\int_0^T\left[f_j(t)-f_j^{(n_k)}(t)\right]\hat{x}_j^{(n_k)}(t)dt+V(P_{n_k}).\quad\text{(by (21))}\end{aligned}$$

This implies

$$\int_0^T\sum_{j=1}^{q}f_j(t)\hat{x}_j^{(n_k)}(t)dt-\sum_{j=1}^{q}\int_0^T\left[f_j(t)-f_j^{(n_k)}(t)\right]\hat{x}_j^{(n_k)}(t)dt=V(P_{n_k}).\tag{41}$$

Besides, since

$$\int_0^T g(t)\widetilde{w}^{(n_k)}(t)dt$$

$$= \int_0^T g(t)\hat{w}^{(n_k)}(t)dt + \int_0^T g(t)\epsilon_{n_k}\rho e^{\rho(T-t)}dt$$

$$= \int_0^T [g(t) - g^{(n_k)}(t)]\hat{w}^{(n_k)}(t)dt + \int_0^T g^{(n_k)}(t)\hat{w}^{(n_k)}(t)dt + \epsilon_{n_k}\int_0^T g(t)\rho e^{\rho(T-t)}dt$$

$$= \int_0^T [g(t) - g^{(n_k)}(t)]\hat{w}^{(n_k)}(t)dt + V(D_{n_k}) + \delta_{2^{n_k}}\int_0^T \rho e^{\rho(T-t)}g^{(n)}(t)dt$$

$$+ \epsilon_{n_k}\int_0^T g(t)\rho e^{\rho(T-t)}dt,$$

we have

$$\int_0^T g(t)\widetilde{w}^{(n_k)}(t)dt - \int_0^T [g(t) - g^{(n_k)}(t)]\hat{w}^{(n_k)}(t)dt$$
$$- \delta_{2^{n_k}}\int_0^T \rho e^{\rho(T-t)}g^{(n_k)}(t)dt - \epsilon_{n_k}\int_0^T g(t)\rho e^{\rho(T-t)}dt = V(D_{n_k}). \tag{42}$$

Hence, by (41) and (42), we obtain

$$\int_0^T \sum_{j=1}^q f_j(t)\hat{x}_j^{(n_k)}(t)dt - \sum_{j=1}^q \int_0^T \left[f_j(t) - f_j^{(n_k)}(t)\right]\hat{x}_j^{(n_k)}(t)dt$$
$$= \int_0^T g(t)\widetilde{w}^{(n_k)}(t)dt - \int_0^T [g(t) - g^{(n_k)}(t)]\hat{w}^{(n_k)}(t)dt$$
$$- \delta_{2^{n_k}}\int_0^T \rho e^{\rho(T-t)}g^{(n_k)}(t)dt - \epsilon_{n_k}\int_0^T g(t)\rho e^{\rho(T-t)}dt. \tag{43}$$

Taking the limit $n_k \to \infty$ on both sides of (43), by (35), (36) and Lebesgue's bounded convergence theorem, we obtain

$$\sum_{j=1}^q \int_0^T f_j(t)\hat{x}_j(t)dt = \int_0^T g(t)\widetilde{w}(t)dt,$$

and this implies

$$\sum_{j=1}^q \int_0^T f_j(t)\hat{x}_j^\star(t)dt = \int_0^T g(t)\widetilde{w}^\star(t)dt,$$

that is, the objective values of $\hat{x}^\star(t)$ and $\widetilde{w}^\star(t)$ are equal. Hence $\hat{x}^\star(t)$ and $\widetilde{w}^\star(t)$ are the optimal solutions of (*SP*) and (*DSP*), respectively. We complete this proof. □

Let $\{\hat{x}^{(n')}(t)\}$ and $\{\widetilde{w}^{(n')}(t)\}$ be any subsequences of $\{\hat{x}^{(n)}(t)\}$ and $\{\widetilde{w}^{(n)}(t)\}$, respectively. Then, by the proof of Theorem 4, there exist weak* convergent subsequences $\{\hat{x}^{(n'_k)}(t)\}$ and $\{\widetilde{w}^{(n'_k)}(t)\}$. Hence if (*SP*) and (*DSP*) have the unique optimal solutions $\hat{x}^\star(t)$ and $\widetilde{w}^\star(t)$, respectively. Then $\{\hat{x}^{(n'_k)}(t)\}$ and $\{\widetilde{w}^{(n'_k)}(t)\}$ are weak* convergent to $\hat{x}^\star(t)$ and $\widetilde{w}^\star(t)$, respectively. In other words, all subsequences of $\{\hat{x}^{(n)}(t)\}$ and $\{\widetilde{w}^{(n)}(t)\}$ have further subsequences that are weak* convergent to $\hat{x}^\star(t)$ and $\widetilde{w}^\star(t)$, respectively. Therefore, we have the following result.

Theorem 5 *Suppose that assumptions* (A1) *and* (A2) *hold. If* (*SP*) *and* (*DSP*) *have the unique optimal solutions* $\hat{x}^\star(t)$ *and* $\widetilde{w}^\star(t)$*, respectively. Then* $\hat{x}^{(n)}(t) \to \hat{x}^\star(t)$ $(weak^*)$ *and* $\widetilde{w}^{(n)}(t) \to \widetilde{w}^\star(t)$ $(weak^*)$ *as* $n \to \infty$*, where* $\hat{x}^{(n)}(t)$ *and* $\widetilde{w}^{(n)}(t)$ *are defined as in* (19) *and* (28)*, respectively.*

5 Numerical examples

Finally, for illustration purpose, we use two examples to implement the improved method and to show the quality of the proposed error bound.

Example 1

$$\begin{aligned}
&\text{maximize} && \int_0^1 ln(t+1/2)x(t)dt \\
&\text{subject to} && 2x(t) - 7\int_0^t x(s)ds \le e^t - 1, \forall t \in [0,1] \\
& && x(t) \in L_+^\infty[0,1].
\end{aligned}$$

Table 1 Approximate value $V_n(SP)$ and error bound ε_n for Examples 1 and 2

n	Example 1		Example 2	
	$V_n(SP)$	ε_n	$V_n(SP)$	ε_n
10	0.3505765	0.0568731	0.8917267	0.2040224
11	0.3518269	0.0285131	0.8958349	0.1025288
12	0.3524547	0.0142757	0.8978995	0.0513945
13	0.3527692	0.0071426	0.8989344	0.0257298
14	0.3529266	0.0035725	0.8994525	0.0128731
15	0.3530054	0.0017866	0.8997118	0.0064386
16	0.3530448	0.0008934	0.8998414	0.0032198
17	0.3530645	0.0004467	0.8999063	0.0016100
18	0.3530743	0.0002234	0.8999387	0.0008050
19	0.3530792	0.0001117	0.8999549	0.0004025
20	0.3530817	0.0000558	0.8999630	0.0002013

Example 2

$$\text{maximize} \quad \int_0^1 \left[(t - 1/2)x_1(t) + (t^2 - 1/3)x_2(t)\right] dt$$

$$\text{subject to} \quad x_1(t) + 3x_2(t) - \int_0^t [4x_1(s) + 2x_2(s)]\, ds \le t, \quad \forall\, t \in [0, 1]$$

$$x_1(t), x_2(t) \in L_+^\infty[0, 1].$$

To illustrate the convergence, we select the partition number n from 10 to 20. Using MATLAB Version 7.0.1 on a PC for the experiment, the results obtained by running the program which implement the proposed algorithm are presented in Table 1, where $V_n(SP)$ is the objective value of the approximate solution $\hat{x}^{(n)}(t)$ and ε_n is the proposed error bound defined as in (34).

From Table 1, one can easily compute a range of the optimal value of (*SP*) for each n, and this range will approach to the optimal value of (*SP*) as n tends to infinite.

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